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# ON SOME CRACK PROBLEMS IN ANISOTROPIC THERMOELASTICITY

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**Abstract**—Some basic equations recently derived by Clements are used to consider crack problems in anisotropic thermoelasticity. The problems concern a single crack in an anisotropic material in which the displacement and stress are independent of one Cartesian coordinate. No symmetry elements of the material are assumed and the temperature, displacement and stress fields are determined for an arbitrary distribution of temperature or heat flux over the crack faces.

## 1. INTRODUCTION

This paper is concerned with deriving for a theory of anisotropic thermoelasticity, results for crack problems which are the counterpart of those given for the isothermal problem by Stroh[1]. Both the equation of heat conduction and the elastic equations are taken to be anisotropic and the analysis holds for general anisotropy in which no symmetry elements of the material are assumed. The uncoupled equations of anisotropic thermoelasticity (Nowacki[2]) are used. Previously, half-space problem (of unmixed type) with temperature specified on the surface of the half-space have been considered by Sharma[3] (for a transversely isotropic medium) by Akoz and Tauchert[4] (for an orthotropic medium) (see also Tauchert and Akoz[5]) and by Clements[6] for more general anisotropy. Here we consider plane crack problems and deduce the effect of the anisotropic thermal and elastic fields when the crack surfaces are subject to either a specified temperature or a temperature gradient.

As might be expected the fully anisotropic situation leads to non-zero stress-singularities in the shear and normal stresses at the crack tip for either of the conditions, temperature or heat flux specified at the crack faces. This is in contrast to the isotropic situation considered by Sih[7] where a specified temperature led to a singularity in the normal stress (mode I fracture) while specified heat flux gave a singularity in the shear stress (mode II fracture). In general, then, in the present problem which is the anisotropic counterpart of the problem of Sih one would get a mixed mode fracture and hence the possibility that the crack would grow not in the direction of the $x_1$ axis but at some angle to it. The determination of this angle would perhaps require higher terms in the stress at the crack tip and the determination of the maximum principle stress near the tip. In Section 6 some illustrative numerical results are presented for a particular transversely anisotropic material.

## 2. FUNDAMENTAL EQUATIONS

Take Cartesian coordinates $x_1$, $x_2$, $x_3$ with $x_2$ vertical and suppose the half-space $x_2 > 0$ is filled with a homogeneous anisotropic elastic material in which the displacement, stress and temperature fields are independent of the Cartesian coordinate $x_3$. The temperature distribution $T(x_1, x_2)$ in the half-space satisfies the heat conduction equation

$$\lambda_{ij}\frac{\partial^2 T}{\partial x_i \partial x_j} = 0, \tag{2.1}$$

where $\lambda_{ij} = \lambda_{ji}$ are the coefficients of heat conduction and the repeated suffix convention (summing from 1 to 3 for Latin suffices only) has been used. For a prescribed temperature

$$T(x_1, 0) = f(x_1), \tag{2.2}$$

on $x_2 = 0$ the temperature in $x_2 > 0$ is given by the expression (see Clements[6])

$$T(x_1, x_2) = \frac{1}{2\pi}\int_0^\infty A^+(p)\exp(ipz') + \bar{A}^+(p)\exp(-ip\bar{z}')\,dp, \tag{2.3}$$

where the bar denotes the complex conjugate and

$$A^+(p) = \int_{-\infty}^\infty f(\xi)\exp(-ip\xi)\,d\xi, \quad \bar{A}^+(p) = \int_{-\infty}^\infty f(\xi)\exp(ip\xi)\,d\xi. \tag{2.4}$$

Also, in (2.3), $z' = x_1 + \tau x_2$ where $\tau$ is the root with positive imaginary part of the quadratic equation

$$\lambda_{11} + 2\lambda_{12}\tau + \lambda_{22}\tau^2 = 0. \tag{2.5}$$

The displacement and stress in the half-space take the form (see Clements[6])

$$u_k = \sum_a A_{ka}\psi_a(z_a) + \sum_a \bar{A}_{ka}\bar{\psi}_a(\bar{z}) + C_k\phi(z') + \bar{C}_k\bar{\phi}(\bar{z}'), \tag{2.6}$$

$$\sigma_{ij} = \sum_a L_{ija}\psi_a'(z_a) + \sum_a \bar{L}_{ija}\bar{\psi}_a'(\bar{z}_a) + N_{ij}\phi'(z') + \bar{N}_{ij}\bar{\phi}'(\bar{z}') - \beta_{ij}T, \tag{2.7}$$

where the sum is from 1 to 3, primes denote derivatives and

$$\phi(z') = \frac{1}{2\pi}\int_0^\infty A^+(p)p^{-1}\exp(ipz')\,dp, \tag{2.8}$$

$$\psi_a(z_a) = \frac{1}{2\pi}\int_0^\infty E_a^+(p)\exp(ipz_a)\,dp, \tag{2.9}$$

where the $E_a^+(p)$ will be chosen subsequently in order to satisfy particular boundary conditions on $x_2 = 0$. In (2.6), the $C_k$ are defined by the equations

$$D_{ik}C_k = \gamma_i, \tag{2.10}$$

where

$$\gamma_i = -i[\beta_{i1} + \tau\beta_{i2}], \tag{2.11}$$

$$D_{ik} = c_{i1k1} + \tau c_{i1k2} + \tau c_{i2k1} + \tau^2 c_{i2k2}, \tag{2.12}$$

$i$ is the square root of minus one, the $\beta_{ij}$ are the stress temperature coefficients and the $c_{ijkl}$ are the elastic constants. Also the $A_{ka}$ are the solutions to the equations

$$(c_{i1k1} + p_a c_{i1k2} + p_a c_{12k1} + p_a^2 c_{i2k2})A_{ka} = 0, \tag{2.13}$$

where the $p_a$ are the roots (with positive imaginary part) of the sextic

$$|c_{i1k1} + pc_{i1k2} + pc_{i2k1} + p^2 c_{i2k2}| = 0. \tag{2.14}$$

The $L_{ija}$ and $N_{ij}$ occurring in eqn (2.7) are defined by

$$L_{ija} = (c_{ijk1} + p_a c_{ijk2})A_{ka}, \tag{2.15}$$

$$N_{ij} = (c_{ijk1} + \tau c_{ijk2})C_k. \tag{2.16}$$

The expressions (2.6)–(2.9) and (2.3) may be used to yield the following expressions for the displacement and stress

$$u_k = \frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_a A_{ka} E_a^+(p) \exp(ipz_a) + C_k A^+(p) p^{-1} \exp(ipz') \right\} \mathrm{d}p, \tag{2.17}$$

$$\sigma_{ij} = \frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_a L_{ija} E_a^+(p) ip \exp(ipz_a) + (iN_{ij} - \beta_{ij}) A^+(p) \exp(ipz') \right\} \mathrm{d}p, \tag{2.18}$$

where $\mathcal{R}$ denotes the real part of a complex number.

If the half-space $x_2 < 0$ is filled with the same material as is in $x_2 > 0$ then the corresponding expressions for the temperature, displacement and stress are

$$T(x_1, x_2) = \frac{1}{2\pi} \int_0^\infty \{A^-(p) \exp(-ipz') + \bar{A}^-(p) \exp(ip\bar{z}')\} \mathrm{d}p, \tag{2.19}$$

$$u_k = \frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_a A_{ka} E_a^-(p) \exp(-ipz_a) - C_k A^-(p) p^{-1} \exp(-ipz')\right\} \mathrm{d}p, \tag{2.20}$$

$$\sigma_{ij} = -\frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_a L_{ija} E_a^-(p) ip \exp(-ipz_a) - (iN_{ij} - \beta_{ij}) A^-(p) \exp(-ipz') \right\} \mathrm{d}p, \tag{2.21}$$

where $A^+(p)$ and $A^-(p)$ are related to a prescribed boundary temperature by expressions of the type given by (2.4). The expressions (2.20) and (2.21) may be obtained by using the same procedure as that employed by Clements[6] to derive the corresponding expressions for the upper half-space.

## 3. THE CRACK PROBLEMS

Consider an infinite anisotropic material in which there is a crack in the region $|x_1| \le a, -\infty < x_3 < \infty$ in the plane $x_2 = 0$. The surface of the crack is subjected to the tractions $\sigma_{i2} = -\tau_i(x_1)$. In addition to these tractions we consider two situations concerning the temperature field. (i) A prescribed temperature $T(x_1, 0) = f(x_1)$ acts on the crack surface. (ii) A prescribed heat flux $\partial T/\partial x_2 = -Q(x_1)$ acts across the crack surface.

It is required to find suitable stress and temperature distributions which satisfy the above boundary conditions at the crack and satisfy certain conditions (to be specified later) at infinity. The corresponding problems in an isotropic medium have been considered by Sih[7].

## 4. TEMPERATURE FIELDS

It is convenient to consider the regions $x_2 > 0$ and $x_2 < 0$ separately. In $x_2 > 0$ the temperature is given by eqn (2.3) while in $x_2 < 0$ the relevant equation is (2.19). The temperature field is of course different for problems (i) and (ii) so we consider these cases separately.

(i) *Specified temperature on crack faces.* The temperature is specified on the crack surface so

$$T(x_1, 0) = f(x_1) \quad \text{for} \quad |x_1| < a. \tag{4.1}$$

The condition that the temperature is continuous on $x_2 = 0$ leads, from (2.3) and (2.19), to the relation

$$A^+(p) = \bar{A}^-(p). \tag{4.2}$$

Also, using (4.2), continuity of heat flux across $x_2 = 0$ for $|x_1| > a$ leads to the condition

$$\int_0^\infty p\{A^+(p) \exp(ipx_1) + \bar{A}^+(p) \exp(-ipx_1)\} \mathrm{d}p = 0 \quad \text{for} \quad |x_1| > a \tag{4.3}$$

and condition (4.1) requires

$$\frac{1}{2\pi}\int_0^\infty \{A^+(p)\exp(ipx_1)+\bar{A}^+(p)\exp(-ipx_1)\}\,dp = f(x_1) \quad \text{for} \quad |x_1|<a. \tag{4.4}$$

We assume that $f(x_1)$ is an even function of $x_1$ so that eqns (4.3) and (4.4) reduce to the pair

$$\int_0^\infty pA^+(p)\cos(px_1)\,dp = 0 \quad \text{for} \quad x_1>a,$$

$$\int_0^\infty A^+(p)\cos(px_1)\,dp = \pi f(x_1) \quad \text{for} \quad 0<x_1<a, \tag{4.5}$$

where $A^+(p)$ is now a real function (the case for general $f(x_1)$ can be treated similarly by writing it as the sum of even and odd functions).

To evaluate the temperature field throughout the body in this problem some care must be taken with the boundary conditions at infinity. In the case when $f(x_1)=f_0$ (a constant) for example (i.e. a constant temperature on the crack faces) the temperature will behave like $\log r$ at infinity and because of this behaviour there is a trivial solution $T=f_0$ to this problem. Also if $T_0(x_1,x_2)$ is a solution so is $T=\alpha T_0(x_1,x_2)+(1-\alpha)f_0$ so that the problem is not uniquely posed. To be specific, consider the case $f(x_1)=f_0$. Then the temperature field can be written as

$$T=\frac{f_0}{\log a}\,\mathcal{R}\,[\log z'+(z'^2-a^2)^{1/2}\}]. \tag{4.6}$$

This satisfies all the boundary conditions on $x_2=0$ but so does

$$T=\alpha T_0+(1-\alpha)f_0 \tag{4.7}$$

for any constant $\alpha$ as mentioned above. To determine the constant $\alpha$ we can consider which three dimensional problem the plane problem consider here is an approximation to. Considerations of this kind have been made by Dharmadesa[8] for certain potential problems. As far as the singular stresses at the crack tip are concerned, however, only the temperature on the crack face is important and the above argument does not affect the stress near the crack tips. This is discussed later in Section 5.

(ii) *Specified heat flux across the crack.*

$$\frac{\partial T}{\partial x_2}=-Q(x_1) \quad \text{on} \quad x_2=0 \quad \text{for} \quad |x_1|<a.$$

Since the heat flux (which is proportional to $\partial T/\partial x_2$) is now specified across the crack and is necessarily continuous across $x_2=0$, $|x_1|>a$ it must be continuous on $x_2=0$ for all $x_1$. Hence from (2.3) and (2.19) we require

$$\tau A^+(p)=\bar{\tau}\bar{A}^-(p)=A_1+iA_2=A(p), \tag{4.8}$$

where $A_1$ and $A_2$ are real functions of $p$. Use of this condition, the specified flux for $|x_1|<a$ and continuity of temperature $T$ for $|x_1|>a$ leads to the equations

$$\mathcal{R}\int_0^\infty\left(\frac{\bar{\tau}-\tau}{\tau\bar{\tau}}\right)A(p)\exp(ipx_1)\,dp=0 \quad \text{for} \quad |x_1|>a, \tag{4.9}$$

$$\frac{1}{\pi}\mathcal{R}\int_0^\infty ipA(p)\exp(ipx_1)\,dp=-Q(x_1) \quad \text{for} \quad |x_1|<a. \tag{4.10}$$

If $Q(x_1)$ is an even function then $A_1=0$ and (4.9) and (4.10) reduce to

$$\int_0^\infty A_2(p)\cos(px_1)\,\mathrm{d}p = 0 \quad \text{for} \quad |x_1| > a, \tag{4.11}$$

$$\int_0^\infty pA_2(p)\cos(px_1)\,\mathrm{d}p = \pi Q(x_1) \quad \text{for} \quad |x_1| < a. \tag{4.12}$$

These integral equations have the solution (see Busbridge[7])

$$A_2(p) = \int_0^a \mu J_0(\mu p)\,\mathrm{d}\mu \int_{-1}^1 Q(\mu\xi)(1-\xi^2)^{-(1/2)}\,\mathrm{d}\xi, \tag{4.13}$$

where, in the usual notation, $J_0$ is the Bessel function of order zero. If $Q(x) = Q_0$ (constant) then (4.13) reduces to

$$A_2(p) = \pi Q_0 \int_0^a \mu J_0(\mu p)\,\mathrm{d}\mu. \tag{4.14}$$

Once $A^+(p)$ and $A^-(p)$ have been calculated from (4.8) and (4.14), the temperature throughout the material may be obtained from (2.3) and (2.19). For example, when $Q(x) = Q_0$ the temperature difference across the crack is given by

$$T = -\frac{2i(\tau-\bar{\tau})}{\tau\bar{\tau}} Q_0(a^2 - x_1^2)^{1/2} \quad \text{for} \quad |x_1| < a. \tag{4.15}$$

## 5. THE STRESS FIELD

As in Section 4 the regions $x_2 > 0$ and $x_2 < 0$ are considered separately. In $x_2 > 0$ the displacement and stress are given by the expressions

$$u_k = \frac{1}{\pi}\mathcal{R}\int_0^\infty \left\{\sum_a A_{ka}E_a^+(p)\exp(ipz_a) + C_kA^+(p)p^{-1}\exp(ipz')\right\}\mathrm{d}p, \tag{5.1}$$

$$\sigma_{ij} = \frac{1}{\pi}\mathcal{R}\int_0^\infty \left\{\sum_a L_{ija}E_a^+(p)ip\exp(ipz_a) + (iN_{ij} - \beta_{ij})A^+(p)\exp(ipz')\right\}\mathrm{d}p \tag{5.2}$$

and in $x_2 < 0$ the corresponding expressions are

$$u_k = \frac{1}{\pi}\mathcal{R}\int_0^\infty \left\{\sum_a A_{ka}E_a^-(p)\exp(-ipz_a) - C_kA^-(p)p^{-1}\exp(-ipz')\right\}\mathrm{d}p, \tag{5.3}$$

$$\sigma_{ij} = -\frac{1}{\pi}\mathcal{R}\int_0^\infty \left\{\sum_a L_{ija}E_a^-(p)ip\exp(-ipz_a) - (iN_{ij} - \beta_{ij})A^-(p)\exp(-ipz')\right\}\mathrm{d}p. \tag{5.4}$$

The requirement that the stresses $\sigma_{i2}$ be continuous on $x_2 = 0$ immediately yields

$$ip\left[\sum_a \{L_{i2a}E_a^+(p) - \bar{L}_{i2a}\bar{E}_a^-(p)\}\right] + i[N_{i2}A^+(p) + \bar{N}_{i2}\bar{A}^-(p)$$
$$-\beta_{i2}[A^+(p) - \bar{A}^-(p)] = 0. \tag{5.5}$$

Equation (5.5) may be rearranged to yield

$$\sum_a L_{i2a}E_a^+(p) + p^{-1}A^+(p)\{N_{i2} + i\beta_{i2}\} = \sum_a \bar{L}_{i2a}\bar{E}_a^-(p) - p^{-1}\bar{A}^-(p)\{\bar{N}_{i2} - i\beta_{i2}\}. \tag{5.6}$$

Denoting these expressions by $F_i(p)$ it follows that

$$E_a^+(p) = M_{ai}F_i(p) - M_{ai}\{N_{i2} + i\beta_{i2}\}A^+(p)p^{-1}, \tag{5.7}$$

$$E_a^-(p) = M_{ai}\bar{F}_i(p) + M_{ai}\{N_{i2} + i\beta_{i2}\}A^-(p)p^{-1}. \tag{5.8}$$

where

$$\sum_a L_{i2\alpha}M_{aj} = \delta_{ij}. \tag{5.9}$$

The traction boundary conditions on the crack surface and the continuity of displacement outside the crack (on $x_2 = 0$) remain to be satisfied. From (5.1) and (5.3) it follows that the displacement will be continuous if

$$\mathcal{R}\int_0^\infty \left[\sum_a \{A_{ka}E_a^+(p) - \bar{A}_{ka}\bar{E}_a^-(p)\} + C_kA^+(p)p^{-1} + \bar{C}_k\bar{A}^-(p)p^{-1}\right]\exp(ipx_1)\,dp = 0 \quad \text{for} \quad |x_1| > a. \tag{5.10}$$

Equations (5.7) and (5.8) may be employed to write (5.10) in the form

$$\mathcal{R}\int_0^\infty [H_{ks}F_s(p) + R_k(p)]\exp(ipx_1)\,dp = 0 \quad \text{for} \quad |x_1| > a, \tag{5.11}$$

where

$$H_{ks} = B_{ks} - \bar{B}_{ks}, \quad B_{ks} = A_{ka}M_{as} \tag{5.12}$$

and

$$R_k(p) = \{-B_{ki}(N_{i2} + i\beta_{i2}) + C_k\}A^+(p)p^{-1} + \{-B_{ki}(\bar{N}_{i2} - i\beta_{i2}) + \bar{C}_k\}\bar{A}^-(p)p^{-1}. \tag{5.13}$$

The condition of prescribed tractions $\sigma_{i2} = p_i(x_1) + q_i(x_1)$ (where $p_i(x_1)$ and $q_i(x_1)$ are, respectively, even and odd functions of $x_1$) over the crack surface will be satisfied if

$$\frac{1}{\pi}\mathcal{R}\int_0^\infty F_j(p)ip\exp(ipx_1)\,dp = p_j(x_1) + q_j(x_1) \quad \text{for} \quad |x_1| < a. \tag{5.14}$$

If we write $F_j = F'_j + iF''_j$ and $R_k = R'_k + iR''_k$ where the primed quantities are real then eqns (5.11) and (5.14) become

$$\left.\begin{aligned} &\int_0^\infty pF'_k(p)\sin(px_1)\,dp = -\pi q_k(x_i) \quad \text{for} \quad |x_1| < a, \\ &\int_0^\infty [iH_{ks}F'_s(p) - R''_k]\sin(px_1)\,dp = 0 \quad \text{for} \quad |x_1| > a \end{aligned}\right\} \tag{5.15}$$

and

$$\left.\begin{aligned} &\int_0^\infty pF''_k(p)\cos(px_1)\,dp = -\pi p_k(x_1) \quad \text{for} \quad |x_1| < a, \\ &\int_0^\infty [iH_{ks}F''_s(p) + R'_k]\cos(px_1)\,dp = 0 \quad \text{for} \quad |x_1| > a. \end{aligned}\right\} \tag{5.16}$$

Stroh[1] has shown that the matrix $H_{ks}$ is non-singular and hence there exists an inverse matrix $T_{jk}$ such that

$$T_{jk}H_{ks} = \delta_{js}. \tag{5.17}$$

Use of (5.17) in (5.15) and (5.16) yields the dual integral equations

$$\left.\begin{aligned}&\int_0^\infty p[F_k'(p)+iT_{ks}R_s''] \sin(px_1)\,\mathrm{d}p = V_k(x_1) \quad \text{for} \quad |x_1|<a,\\ &\int_0^\infty [F_k'(p)+iT_{ks}R_s''] \sin(px_1)\,\mathrm{d}p = 0 \quad \text{for} \quad |x_1|>a\end{aligned}\right\} \tag{5.18}$$

and

$$\left.\begin{aligned}&\int_0^\infty p[F_k''(p)-iT_{ks}R_s'] \cos(px_1)\,\mathrm{d}p = W_k(x_1) \quad \text{for} \quad |x_1|<a,\\ &\int_0^\infty [F_k''(p)-iT_{ks}R_s'] \cos(px_1)\,\mathrm{d}p = 0 \quad \text{for} \quad |x_1|>a,\end{aligned}\right\} \tag{5.19}$$

where

$$V_k(x_1) = -\pi q_k(x_1) + \int_0^\infty ipT_{ks}R_s'' \sin(px_1)\,\mathrm{d}p, \tag{5.20}$$

$$W_k(x_1) = -\pi p_k(x_1) - \int_0^\infty ipT_{ks}R_s' \cos(px_1)\,\mathrm{d}p. \tag{5.21}$$

The general solution of (5.18) and (5.19) is (see Stroh[1])

$$F_k'(p)+iT_{ks}R_s'' = \frac{1}{\pi}\int_0^a \mu J_1(\mu p)\,\mathrm{d}\mu \int_{-1}^1 V_k(\mu\xi)(1-\xi^2)^{-(1/2)}\xi\,\mathrm{d}\xi, \tag{5.22}$$

$$F_k''(p)-iT_{ks}R_s' = \frac{1}{\pi}\int_0^a \mu J_0(\mu p)\,\mathrm{d}\mu \int_{-1}^1 W_k(\mu\xi)(1-\xi^2)^{-(1/2)}\,\mathrm{d}\xi. \tag{5.23}$$

Once the heat flux and tractions are known the displacement and stress throughout the material may be found through (5.22), (5.23), (5.8) and (5.1)–(5.4).

However, to evaluate the singular stress field at the crack tip all one needs to do is to evaluate the effective loading induced on the crack via the temperature field. This can be done from (5.20) and (5.21) and then results for the near crack tip stress field follow from the results derived by Stroh[1] for the elastic case. To justify this we need to show that the expressions $iT_{ks}R_s''$ and $iT_{ks}R_s'$ on the left hand sides of (5.22) and (5.23) do not contribute to the singular stress field. This is relatively easy to show and is demonstrated below. We discuss the effect of the two temperature fields outlined in Section 4.

(i) *Specified temperature on the crack faces*

Restricting attention to the situation when the temperature specified on the crack $f(x_1)$ is an even function of $x_1$ we have that $A^+(p)$ is a real function of $p$. Hence, using (4.2), (5.13) becomes

$$R_k(p) = 2A^+(p)p^{-1}\mathcal{R}\{C_k - B_{ki}(N_{i2}+i\beta_{i2})\} \tag{5.24}$$

so $R_k''(p)=0$ and $R_k(p)\equiv R_k'(p)$. Thus, in (5.20) there is no contribution to $V_k(x_1)$ from the temperature field and in (5.21) we have

$$W_k(x_1) = -\pi p_k(x_1) - 2iT_{ks}\mathcal{R}\{C_s - B_{si}(N_{i2}+i\beta_{i2})\}\int_0^\infty A^+(p)\cos(px_1)\,\mathrm{d}p$$

and the integral is just the temperature which is specified on the crack faces. Hence, from (4.5) we get

$$W_k(x_1) = -\pi\{p_k(x_1) + 2if(x_1)T_{ks}\mathcal{R}[C_s - B_{si}(N_{i2} + i\beta_{i2})]\}. \tag{5.25}$$

From (5.2) and (5.7) the stress on $x_2 = 0+$ can be written

$$\begin{aligned}\sigma_{s2} &= \pi^{-1}\mathcal{R}\int_0^\infty ipF_s(p)\exp(ipx_1)\,\mathrm{d}p \\ &= -\pi^{-1}\int_0^\infty p[F'_s \sin px_1 + F''_s \cos px_1]\,\mathrm{d}p\end{aligned} \tag{5.26}$$

and from (5.24) it can be seen that the effect of the terms $iT_{ks}R'_s$ of (5.23) is just to give a stress which is proportional to the temperature field and since the temperature field cannot be singular at the crack tip this term gives rise to a non-singular stress at the crack tip. Note, however, that the influence of the temperature on $W_k(x_1)$ (eqn (5.25) above), does affect the singular crack tip stress field. Thus, for the purposes of determining the singular stress at the crack tip it is only necessary to know the temperature distribution over the faces of the crack. However, to calculate the stress throughout the material it is necessary to obtain the complete temperature field.

In the case of a stress free crack with a given constant applied temperature $f(x_1) = f_0$ (constant) eqn (5.25) yields

$$W_k(x_1) = \mathcal{W}_k \text{ (constant)} \tag{5.27}$$

where

$$\mathcal{W}_k = -2i\pi f_0 T_{ks}\mathcal{R}[C_s - B_{si}(N_{i2} + i\beta_{i2})]. \tag{5.28}$$

Hence, from (5.23) and (5.26) we obtain an expression for the stresses on $x_2 = 0$ near the crack tip at $x_1 = a$ in the form

$$\sigma_{k2} \simeq \mathcal{W}_k \pi^{-1}(a/2r)^{1/2} \tag{5.29}$$

where $r = x_1 - a$ with $x_1 > a$.

(ii) *Specified heat flux across the crack*

As in (ii), Section 4, we consider the heat flux $Q(x_1)$ to be a symmetric function of $x_1$. Then $A(p) = iA_2(p)$ is a purely imaginary function of $p$ and (5.13) becomes

$$R_k(p) = 2iA_2(p)p^{-1}\mathcal{R}\{\tau^{-1}[C_k - B_{ki}(N_{i2} + i\beta_{i2})]\} \tag{5.30}$$

giving $R'_k = 0$ and $R_k(p) = iR''_k(p)$. For this problem there is no contribution to $W_k(x_1)$ from the temperature field but (5.20) gives

$$V_k(x_1) = -\pi q_k(x_1) + 2iT_{ks}\mathcal{R}\{\tau^{-1}[C_s - B_{si}(N_{i2} + i\beta_{i2})]\}\int_0^\infty A_2(p)\sin(px_1)\,\mathrm{d}p. \tag{5.31}$$

Integrating eqn (4.12) with respect to $x_1$, we obtain

$$\int_0^\infty A_2(p)\sin px_1\,\mathrm{d}p = \pi\int_0^{x_1} Q(t)\,\mathrm{d}t \quad \text{for} \quad 0 \leq x_1 < a. \tag{5.32}$$

When the applied tractions are zero and the heat flux is constant over the crack faces ($Q(x_1) = Q_0$, say) then

$$V_k(x_1) = \mathcal{V}_k x_1 \tag{5.33}$$

where

$$\mathcal{V}_k = 2iT_{ks}Q_0\mathcal{R}\{\tau^{-1}[C_s - B_{si}(N_{i2} + i\beta_{i2})]\}. \tag{5.34}$$

Hence, from (5.22) and (5.26) we obtain an expression for the stresses on $x_2 = 0$ near the crack tip at $x_1 = a$ in the form

$$\sigma_{k2} \simeq \frac{\mathcal{V}_k}{2\pi}\frac{a^{3/2}}{r^{1/2}}. \tag{5.35}$$

## 6. NUMERICAL RESULTS

In this section we consider the stress near the crack tip in a particular transversely isotropic material.

For transversely isotropic materials with the $x_1$ and $x_2$-axes lying in the transverse plane the non-zero $c_{ijkl}$, $\alpha_{ij}$ and $\lambda_{ij}$ are

$$c_{1111} = c_{2222},\ c_{1133} = c_{2233},\ c_{1313} = c_{2323},\ c_{1122},$$
$$c_{3333},\ c_{1212} = (c_{1111} - c_{1122}),\ \alpha_{11} = \alpha_{22},\ \alpha_{33},$$
$$\lambda_{11} = \lambda_{22},\ \lambda_{33}.$$

If a rotation of $\alpha$ about the $x_2$-axis is followed by a rotation of $\theta$ about the $x_1$-axis then the constants referred to the rotated frame are given by

$$c'_{ijkl} = a_{im}a_{jn}a_{kp}a_{lq}c_{mnpq},\quad \beta'_{ij} = a_{im}a_{jn}\beta_{mn},\quad \lambda'_{ij} = a_{im}a_{jn}\lambda_{mn},$$

where

$$[a_{ij}] = \begin{bmatrix} \cos\alpha & -\sin\alpha\sin\theta & -\sin\alpha\cos\theta \\ 0 & \cos\theta & \sin\theta \\ -\sin\alpha & -\cos\alpha\sin\theta & \cos\alpha\cos\theta \end{bmatrix}$$

We consider the particular transversely isotropic material which, referred to the symmetry axes with the $x_3$-axis normal to the transverse plane, has constants

$$c_{1111} = 16.5,\ c_{1122} = 3.1,\ c_{1133} = 5,\ c_{3333} = 6.2,$$
$$c_{1313} = 3.92,\ 10^6\alpha_{11} = 60.8,\ 10^6\alpha_{33} = 14.3,\ \lambda_{11}/\lambda_{33} = 1.17$$

If the elastic constants are multiplied by $10^{11}$ then the units for these constants are dynes/cm$^2$ while the coefficients of linear thermal expansion are for a temperature increase of one degree centigrade. These are the values of the material constants for a crystal of zinc although they are chosen here merely for illustrative purposes. The values of $\mathcal{W}_k$ and $\mathcal{V}_k$ for such a material for various combinations of $\alpha$ and $\theta$ are listed in Table 1.

Table 1.

| $\alpha$ | 0 | 0 | $\pi/4$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|---|---|---|---|---|---|---|
| $\theta$ | $\pi/2$ | $\pi/4$ | 0 | $\pi/4$ | $\pi/6$ | 0 |
| $\mathcal{W}_1$ | 0 | 0 | 0 | 4.73 | 4.69 | 0 |
| $\mathcal{W}_2$ | −2.14 | −3.68 | −2.09 | −4.99 | −4.99 | −3.47 |
| $\mathcal{W}_3$ | 0 | 1.84 | 0 | −1.3 | −2.26 | 0 |
| $\mathcal{V}_1$ | 0.94 | 1.37 | 1.12 | −0.23 | −0.42 | 0.8 |
| $\mathcal{V}_2$ | 0 | 0 | 0 | 0.71 | 0.84 | 0 |
| $\mathcal{V}_3$ | 0 | 0 | −1.9 | −0.3 | −0.19 | 0 |

If each of the constants occurring in Table 1 are multiplied by $10^7$ then the units for the constants are dynes/cm$^2$.

We now use the analysis of Section 5 together with the results in Table 1 to consider the singular nature of the stress in the plane of the crack near to the crack tip at $x_1 = a$.

*Case* 1: $\alpha = 0$, $\theta = \pi/2$. Each of the planes $x = 0$, $i = 1, 2, 3$ is a plane of elastic symmetry.

For a constant applied temperature $f_0$ it is apparent from Table 1 and eqn (5.29) that the shear stresses $\sigma_{12}$ and $\sigma_{23}$ are not singular in the plane of the crack.

For a constant heat flux $Q_0$ the stress $\sigma_{12}$ is singular in the plane of the crack but $\sigma_{22}$ and $\sigma_{23}$ do not exhibit singular behaviour.

These results are consistent with those previously obtained by Sih[5] for the corresponding problems for isotropic materials.

*Case* 2: $\alpha = 0$, $\theta = \pi/4$. The $x_1 = 0$ plane is a plane of elastic symmetry while the $x_2 = 0$ and $x_3 = 0$ planes are not planes of elastic symmetry.

For a constant applied temperature $f_0$ the stress $\sigma_{12}$ is not singular in the plane of the crack but both the $\sigma_{22}$ and $\sigma_{23}$ stresses exhibit singular behaviour.

For a constant heat flux $Q_0$ the stress $\sigma_{12}$ is singular in the plane of the crack but $\sigma_{22}$ and $\sigma_{23}$ do not exhibit singular behaviour.

*Case* 3: $\alpha = \pi/4$, $\theta = 0$. The $x_2 = 0$ plane is a plane of elastic symmetry, while the $x_1 = 0$ and $x_3 = 0$ planes are not planes of elastic symmetry.

For a constant applied temperature $f_0$ the $\sigma_{22}$ stress is singular on $x_2 = 0$ but the $\sigma_{12}$ and $\sigma_{23}$ stresses are not singular.

For a constant heat flux $Q_0$ the $\sigma_{12}$ and $\sigma_{23}$ stresses are singular but the $\sigma_{22}$ stress is not singular on $x_2 = 0$. It is of interest to note that, for the cases considered, this particular combination of angles gives the largest singular stress near the crack tip for a given constant heat flux $W_0$.

*Case* 4: $\alpha = \pi/4$, $\theta = \pi/4$ and $\alpha = \pi/3$, $\theta = \pi/6$. None of the planes $x_i = 0$, $i = 1, 2, 3$ are planes of elastic symmetry in this case. All the stresses $\sigma_{12}$, $i = 1, 2, 3$ are singular on $x_2 = 0$ for both an applied temperature and a given heat flux. These combinations of angles give a markedly larger singular stress on $x_2 = 0$ for a given applied temperature $f_0$ than any of the other angles considered.

*Case* 5: $\alpha = \pi/2$, $\theta = 0$. All of the planes $x_i = 0$, $i = 1, 2, 3$ are planes of elastic symmetry. The stress pattern is similar to that outlined for case 1.

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